

A GENERALIZATION OF INVARIANT RIEMANNIAN METRICS AND ITS APPLICATIONS TO RICCI SOLITONS

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ABSTRACT. In this paper we introduce a generalization of the concept of invariant Riemannian metric. This new definition can help us to find Riemannian manifolds with special geometric properties. As an application, by this method, we give many explicit examples of shrinking, steady and expanding Ricci solitons.

1. Introduction

The geometry of Lie groups and homogeneous spaces equipped with invariant Riemannian metrics is a basic part of Riemannian geometry which have been considered by many great mathematicians (see [1], [2], [3], [5], [7], [9]). This importance comes from the simplicity of the structures of these spaces (versus the general case) and their applications in other fields such as theoretical physics. Although, mathematicians have found many interesting and important applications of these spaces but it seems that such spaces are too special, because the study of their geometry is limited to the study of an inner product on a vector space (Lie algebra). For example anyone knows that every commutative Lie group equipped with a left invariant Riemannian metric is flat, but one may want to work on a commutative Lie group such that its geometric structures have an interaction with its group structure and is not flat. In the present article we give a very simple generalization of invariant Riemannian metrics on Lie groups such that the inner product on any tangent space is controlled by a smooth function which varies on the base manifold. At the first look it seems that it is a slight generalization, but we can see by this method we can construct many examples of Riemannian manifolds with special curvature properties or other applications. For example by this generalization we construct many different examples of shrinking, steady and expanding Ricci solitons, which show the significance of our definition.

A Ricci soliton with expansion constant λ is a Riemannian manifold (M, g) together with a smooth vector field X on M such the following equation holds;

$$(1.1) \quad \mathcal{L}_X g = 2(\lambda g - \text{Ric}(g)),$$

where $\mathcal{L}_X g$ denotes the Lie derivative of g with respect to X . For a Ricci soliton the Ricci flow equation

$$(1.2) \quad \frac{d}{dt} g_t = -2\text{Ric}(g_t)$$

with initial condition $g_0 = g$ has the solution

$$(1.3) \quad g_t = (1 - 2\lambda t)\phi_t^* g,$$

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in the t -interval on which $1 - 2\lambda t > 0$, where $\phi_t : M \rightarrow M$ is the time t flow of X .

The Ricci soliton is called shrinking if $\lambda > 0$, steady if $\lambda = 0$ and expanding if $\lambda < 0$. If $X = \text{grad}\Phi$ for some smooth real-valued function Φ on M , then the Ricci soliton is called gradient and the function Φ is called a potential function of the Ricci soliton.

In the recent years the geometry of homogeneous Ricci solitons is studied by some mathematicians (for example see [4] and [6]). It is shown that any shrinking or steady homogeneous Ricci soliton must be either a compact Einstein manifold or the product of a compact Einstein manifold with Euclidean space (see [4], [8] and [11]). On the other hand all known examples of expanding homogeneous Ricci solitons are isometric to algebraic Ricci soliton metrics on solvable Lie groups (for more details see [4] and [6]).

In this paper by using the concept f -invariant Riemannian metric on Lie groups we give many examples of shrinking, steady and expanding Ricci solitons which are constructed on (solvable) Lie groups. The other benefit of our method is that anyone, by using a software (such as Maple) and our method, can find new examples of Ricci solitons with different properties.

2. f -invariant Riemannian metrics

In this paper, the statement Riemannian metric can be replaced with the statement semi-Riemannian.

Definition 2.1. Let f be a smooth real positive function on a Lie group G such that $f(e) = 1$, where e is the unit element of G . Suppose that $\langle \cdot, \cdot \rangle$ is a Riemannian metric on G . The metric $\langle \cdot, \cdot \rangle$ is said to be f -left invariant if

$$(2.1) \quad L_b^* \langle X_a, Y_a \rangle_a := \langle L_{b*} X_a, L_{b*} Y_a \rangle_{ba} = \langle X_a, Y_a \rangle_a \cdot \frac{f(ba)}{f(a)},$$

for all $a, b \in G$. Similarly, the Riemannian metric $\langle \cdot, \cdot \rangle$ is said to be f -right invariant if

$$(2.2) \quad R_b^* \langle X_a, Y_a \rangle_a := \langle R_{b*} X_a, R_{b*} Y_a \rangle_{ba} = \langle X_a, Y_a \rangle_a \cdot \frac{f(ab)}{f(a)},$$

for all $a, b \in G$.

Definition 2.2. A Riemannian metric which is both f -left invariant and f -right invariant is called f -bi-invariant. In this case we can see f must be symmetric which means that $f(ab) = f(ba)$, for all $a, b \in G$.

Theorem 2.3. Let f be a smooth positive real function on a Lie group G , such that $f(e) = 1$. Then G admits a f -left invariant Riemannian metric (f -right invariant Riemannian metric).

Proof. Suppose that $\langle \cdot, \cdot \rangle_e$ is any inner product on $T_e G$. For any $a \in G$ let

$$(2.3) \quad \langle X_a, Y_a \rangle_a := f(a) \langle L_{a^{-1}*} X_a, L_{a^{-1}*} Y_a \rangle_e.$$

Notice that $\langle \cdot, \cdot \rangle$ is smooth on TG because the multiplication of G , and f are smooth. Also $\langle \cdot, \cdot \rangle$ is f -left invariant because

$$(2.4) \quad \begin{aligned} L_b^* \langle X_a, Y_a \rangle_a &= \langle L_{b*} L_{a*} (L_{a^{-1}*} X_a), L_{b*} L_{a*} (L_{a^{-1}*} Y_a) \rangle_{ba} \\ &= \langle L_{ba*} (L_{a^{-1}*} X_a), L_{ba*} (L_{a^{-1}*} Y_a) \rangle_{ba} \\ &= f(ba) \langle L_{a^{-1}*} X_a, L_{a^{-1}*} Y_a \rangle_e = \frac{f(ba)}{f(a)} \langle X_a, Y_a \rangle_a. \end{aligned}$$

The proof of the f -right invariant case is exactly similar. \square

Theorem 2.4. *Let $f : G \longrightarrow (\mathbb{R}^+, \cdot)$ be a homomorphism of Lie groups. The Lie group G has a f -bi-invariant Riemannian metric if and only if $\mathfrak{g} = T_e G$ has an inner product which is Ad -invariant.*

Proof. Let $\langle \cdot, \cdot \rangle$ be a f -bi-invariant Riemannian metric on the Lie group G . Suppose that $\langle \cdot, \cdot \rangle_e$ is the inner product induced by $\langle \cdot, \cdot \rangle$ on $T_e G$.

$$\begin{aligned}
 (2.5) \quad \langle Ad_a X_e, Ad_a Y_e \rangle_e &= R_{a^{-1}}^* \langle L_{a*} X_e, L_{a*} Y_e \rangle_a \\
 &= f(a) \langle R_{a^{-1}*} X_e, R_{a^{-1}*} Y_e \rangle_{a^{-1}} \\
 &= f(a) f(a^{-1}) \langle X_e, Y_e \rangle_e = \langle X_e, Y_e \rangle_e.
 \end{aligned}$$

Therefore $\langle \cdot, \cdot \rangle_e$ is Ad -invariant.

Conversely, let $\langle \cdot, \cdot \rangle_e$ be an Ad -invariant inner product on $T_e G$. Now we define a Riemannian metric on G as follows

$$(2.6) \quad \langle X_a, Y_a \rangle_a := f(a) \langle L_{a^{-1}*} X_a, L_{a^{-1}*} Y_a \rangle_e,$$

obviously $\langle \cdot, \cdot \rangle$ is a f -left invariant Riemannian metric on G .

Also it is f -right invariant because

$$\begin{aligned}
 (2.7) \quad R_b^* \langle X_a, Y_a \rangle_a &= R_b^* \langle L_{a*} (L_{a^{-1}*} X_a), L_{a*} (L_{a^{-1}*} Y_a) \rangle_a \\
 &= R_b^* L_a^* \langle L_{a^{-1}*} X_a, L_{a^{-1}*} Y_a \rangle_e \\
 &= (L_a R_b)^* (L_b R_{b^{-1}})^* \langle L_{a^{-1}*} X_a, L_{a^{-1}*} Y_a \rangle_e \\
 &= (L_b R_{b^{-1}} R_b L_a)^* \langle L_{a^{-1}*} X_a, L_{a^{-1}*} Y_a \rangle_e \\
 &= L_a^* L_b^* \langle L_{a^{-1}*} X_a, L_{a^{-1}*} Y_a \rangle_e \\
 &= f(b) f(a) \langle L_{a^{-1}*} X_a, L_{a^{-1}*} Y_a \rangle_e \\
 &= f(b) \langle X_a, Y_a \rangle_a = \frac{f(ab)}{f(a)} \langle X_a, Y_a \rangle_a.
 \end{aligned}$$

\square

Remark 2.5. There is not any nontrivial f -bi-invariant Riemannian metric on a compact Lie group G such that $f : G \longrightarrow (\mathbb{R}^+, \cdot)$ is a homomorphism of Lie groups.

Theorem 2.6. *Let G be a connected Lie group equipped with a f -left invariant Riemannian metric $\langle \cdot, \cdot \rangle$. Then the following are equivalent:*

- (1) $\langle \cdot, \cdot \rangle$ is f -right invariant, hence f -bi-invariant.
- (2) $\langle \cdot, \cdot \rangle$ is $Ad(G)$ -invariant.
- (3) $f(a) \langle \zeta_* X_a, \zeta_* Y_a \rangle_{a^{-1}} = f(a^{-1}) \langle X_a, Y_a \rangle_a$, for all $a \in G$, where ζ is the inversion map.
- (4) $\langle X, [Y, Z] \rangle = \langle [X, Y], Z \rangle$, for all $X, Y, Z \in \mathfrak{g}$.

Proof. (1) \longrightarrow (2)

$$\begin{aligned}
 \langle Ad_a X_e, Ad_a Y_e \rangle_e &= \langle L_{a*} R_{a^{-1}*} X_e, L_{a*} R_{a^{-1}*} Y_e \rangle_e \\
 (2.8) \qquad &= \langle R_{a^{-1}*} X_e, R_{a^{-1}*} Y_e \rangle_{a^{-1}} \frac{f(aa^{-1})}{f(a^{-1})} \\
 &= \langle X_e, Y_e \rangle_e \frac{f(ea^{-1})}{f(e)} \frac{f(e)}{f(a^{-1})} = \langle X_e, Y_e \rangle_e.
 \end{aligned}$$

(2) \longrightarrow (1)

$$\begin{aligned}
 \langle R_{a^{-1}*} X_e, R_{a^{-1}*} Y_e \rangle_{a^{-1}} &= \langle L_{a*} R_{a^{-1}*} X_e, L_{a*} R_{a^{-1}*} Y_e \rangle_e \frac{f(a^{-1})}{f(aa^{-1})} \\
 (2.9) \qquad &= \langle Ad_a X_e, Ad_a Y_e \rangle_e f(a^{-1}) \\
 &= \langle X_e, Y_e \rangle_e f(a^{-1}).
 \end{aligned}$$

(1) \longrightarrow (3)

$$\begin{aligned}
 \langle \zeta_* X_a, \zeta_* Y_a \rangle_{a^{-1}} &= \langle R_{a^{-1}*} \zeta_* L_{a^{-1}*} X_a, R_{a^{-1}*} \zeta_* L_{a^{-1}*} Y_a \rangle_{a^{-1}} \\
 &= \langle \zeta_* L_{a^{-1}*} X_a, \zeta_* L_{a^{-1}*} Y_a \rangle_e f(a^{-1}) \\
 (2.10) \qquad &= \langle -L_{a^{-1}*} X_a, -L_{a^{-1}*} Y_a \rangle_e f(a^{-1}) \\
 &= \langle X_a, Y_a \rangle_a \frac{f(a^{-1})}{f(a)}
 \end{aligned}$$

(3) \longrightarrow (1)

$$\begin{aligned}
 \langle R_{a*} X_e, R_{a*} Y_e \rangle &= \langle \zeta_* L_{a^{-1}*} \zeta_* X_e, \zeta_* L_{a^{-1}*} \zeta_* Y_e \rangle_a \\
 (2.11) \qquad &= \langle L_{a^{-1}*} \zeta_* X_e, L_{a^{-1}*} \zeta_* Y_e \rangle_{a^{-1}} \frac{f(a)}{f(a^{-1})} \\
 &= \langle \zeta_* X_e, \zeta_* Y_e \rangle \frac{f(a^{-1}e)}{f(e)} \frac{f(a)}{f(a^{-1})} \\
 &= \langle X_e, Y_e \rangle_e f(a).
 \end{aligned}$$

(2) \longleftrightarrow (4)

The proof is similar to invariant Riemannian metric case (see Lemma 3 page 302 of [10]). \square

Corollary 2.7. *Let f be a smooth positive real function on a compact Lie group G , such that $f(e) = 1$. Then G admits a f -bi-invariant Riemannian metric.*

3. Curvature of f -invariant Riemannian metrics

In this article we use the notation $\{E_1, \dots, E_n\}$ for a set of left invariant vector fields on a Lie group G which is an orthogonal basis at any point of G and is an orthonormal basis at the unit element e , with respect to a f -left invariant Riemannian metric $\langle \cdot, \cdot \rangle$.

Theorem 3.1. *Let G be a Lie group equipped with a f -left invariant Riemannian metric $\langle \cdot, \cdot \rangle$. Suppose that α_{ijk} are structure constants defined by $[E_i, E_j] = \sum_{k=1}^n \alpha_{ijk} E_k$. Then*

the sectional curvature $K(E_p, E_q)$ is given by the following formula:

$$\begin{aligned}
 K(E_p, E_q) &= \frac{1}{4f^3} \left(2f\delta_{pq}(f_{qp} + f_{pq}) - 2f(f_{pp} + f_{qq}) - 4\delta_{pq}f_p f_q + 2f_p^2 + 2f_q^2 \right. \\
 (3.1) \quad &+ \sum_{r=1}^n (2\delta_{rq}f_q - f_r + 2f\alpha_{rq})(f_r + 2f\delta_{prp}) \\
 &\quad - (\delta_{rp}f_q + f\alpha_{pqr})^2 + (\delta_{qr}f_p - \delta_{pq}f_r + f(\alpha_{rpq} - \alpha_{qrp}))^2 \\
 &\quad \left. - 2f\alpha_{pqr}(\delta_{pr}f_q - \delta_{rq}f_p + f(\alpha_{rqp} + \alpha_{prq} - \alpha_{qpr})) \right),
 \end{aligned}$$

where $f_i := E_i f$ and $f_{ij} := E_j E_i f$.

Proof. The relation $[E_i, E_j] = \sum_{k=1}^n \alpha_{ijk} E_k$ shows that

$$(3.2) \quad \alpha_{ijk} = \frac{1}{f(a)} < [E_i, E_j], E_k >_a.$$

Therefore we have

$$\begin{aligned}
 2 < \nabla_{E_i} E_j, E_k > &= E_i < E_j, E_k > + E_j < E_i, E_k > - E_k < E_i, E_j > \\
 (3.3) \quad &- < E_i, [E_j, E_k] > + < E_j, [E_k, E_i] > + < E_k, [E_i, E_j] > \\
 &= \delta_{jk}f_i + \delta_{ik}f_j - \delta_{ij}f_k + f(-\alpha_{jki} + \alpha_{kij} + \alpha_{ijk}),
 \end{aligned}$$

and so,

$$(3.4) \quad \nabla_{E_i} E_j = \frac{1}{2} \sum_{k=1}^n (\delta_{jk}f_i + \delta_{ik}f_j - \delta_{ij}f_k + f(\alpha_{ijk} + \alpha_{kij} - \alpha_{jki})) E_k.$$

Now for the curvature tensor we have:

$$\begin{aligned}
 R(E_i, E_j)E_k &= \frac{1}{4f^2} \sum_{l=1}^n \left\{ 2f(\delta_{lj}f_{ki} - \delta_{jk}f_{li} - \delta_{li}f_{kj} + \delta_{ik}f_{lj}) \right. \\
 (3.5) \quad &- 2f_i(\delta_{lj}f_k - \delta_{jk}f_l) + 2f_j(\delta_{li}f_k - \delta_{ik}f_l) \\
 &+ \sum_{r=1}^n (\delta_{kr}f_j + \delta_{rj}f_k - \delta_{jk}f_r + f(\alpha_{jkr} + \alpha_{rjk} - \alpha_{krj})) \\
 &\quad \times (\delta_{rl}f_i + \delta_{li}f_r - \delta_{ir}f_l + f(\alpha_{irl} + \alpha_{lir} - \alpha_{rli})) \\
 &\quad - (\delta_{kr}f_i + \delta_{ri}f_k - \delta_{ik}f_r + f(\alpha_{ikr} + \alpha_{rik} - \alpha_{kri})) \\
 &\quad \times (\delta_{rl}f_j + \delta_{lj}f_r - \delta_{jr}f_l + f(\alpha_{jrl} + \alpha_{ljr} - \alpha_{rlj})) \\
 &\quad \left. - 2f\alpha_{ijr}(\delta_{lr}f_k - \delta_{rk}f_l + f(\alpha_{rkl} + \alpha_{lrk} - \alpha_{klr})) \right\} E_l.
 \end{aligned}$$

On the other hand,

$$(3.6) \quad < E_p, E_p > < E_q, E_q > - < E_p, E_q >^2 = f^2.$$

Now the formula of sectional curvature completes the proof. \square

Remark 3.2. If we consider the left invariant Riemannian metrics then the formula given in Theorem (3.1) for the sectional curvature reduces to Milnor's formula given in [7]. It is sufficient to consider f is the constant function $f \equiv 1$.

Remark 3.3. If the Riemannian metric $\langle \cdot, \cdot \rangle$ is f -bi-invariant then the array α_{ijk} is skew in the last two indices for any i . Therefore in this case we have a simpler formula for sectional curvature.

Theorem 3.4. *Let G be a commutative Lie group equipped with a f -left invariant Riemannian metric $\langle \cdot, \cdot \rangle$. Then for the sectional curvature we have*

$$(3.7) \quad K(E_p, E_q) = \frac{1}{2f^3} \left(-f(f_{pp} + f_{qq}) + \frac{3}{2}(f_p^2 + f_q^2) - \frac{1}{2} \sum_{l=1}^n f_l^2 \right).$$

Proof. In formula (3.1) it is sufficient to consider all structure constants are zero. \square

Theorem 3.5. *Consider the assumptions of theorem (3.1). Then the Ricci curvature tensor Ric is given by the following formula:*

$$(3.8) \quad \begin{aligned} Ric(E_p, E_q) = & \frac{1}{4f^2} \sum_{j=1}^n \left(2f(\delta_{jp}f_{qj} - \delta_{pq}f_{jj} - f_{qp} + \delta_{jq}f_{jp}) - 2f_j(\delta_{jp}f_q - \delta_{pq}f_j) + 2f_p(f_q - \delta_{jq}f_j) \right. \\ & + \sum_{r=1}^n (\delta_{qr}f_p + \delta_{rp}f_q - \delta_{pq}f_r + f(\alpha_{pqr} + \alpha_{rpq} - \alpha_{qrp}))(f_r + 2f\alpha_{jrp}) \\ & - (\delta_{qr}f_j + \delta_{rj}f_q - \delta_{jq}f_r + f(\alpha_{jqr} + \alpha_{rjq} - \alpha_{qjr})) \\ & \times (\delta_{rj}f_p + \delta_{jp}f_r - \delta_{pr}f_j + f(\alpha_{prj} + \alpha_{jpr} - \alpha_{rjp})) \\ & \left. - 2f\alpha_{jpr}(\delta_{jr}f_q - \delta_{rq}f_j + f(\alpha_{rqj} + \alpha_{jrq} - \alpha_{qjr})) \right). \end{aligned}$$

Proof. Assume that $e_i := \frac{E_i}{\sqrt{f}}$, for $i = 1, \dots, n$. Note that, in general case, the vector fields e_i are not left invariant. We can see the set $\{e_1, \dots, e_n\}$ is an orthonormal basis at every point of G , with respect to the f -left invariant metric. Now the equations

$$(3.9) \quad Ric(E_p, E_q) = fRic(e_p, e_q) = f \sum_{j=1}^n \langle R(e_j, e_p)e_q, e_j \rangle = \frac{1}{f} \sum_{j=1}^n \langle R(E_j, E_p)E_q, E_j \rangle,$$

together with the formula (3.5) complete the proof. \square

Remark 3.6. In general case, if f is a positive function on an open set M containing $e \in G$ such that $f(e) = 1$ then all previous results are true for the Riemannian manifold M .

4. Applications to constructing Ricci Solitons

In this section, by using the concept f -left invariant Riemannian metric, we give some examples of Ricci solitons constructed on some solvable Lie groups in dimensions two to four. In each case we show that the equation (1.1) reduces to a system of equations which is more simpler than (1.1).

4.1. Lie group $G = \mathbb{R}^2$. Suppose that the Lie group $G = \mathbb{R}^2$ equipped with a f -left invariant Riemannian metric g such that the set $\{E_1 := \frac{\partial}{\partial x}, E_2 := \frac{\partial}{\partial y}\}$ is an orthogonal basis at any point and is orthonormal at $e = (0, 0)$. Suppose that $X = \theta \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}$ is an arbitrary vector

field on G , where θ and η are smooth real functions on G . Then easily we can see the equation (1.1) reduces to the following system of three equations,

$$(4.1) \quad \begin{cases} \theta f_x + \eta f_y + 2f\theta_x = 2(\lambda - \kappa)f \\ \theta f_x + \eta f_y + 2f\eta_y = 2(\lambda - \kappa)f \\ \eta_x = -\theta_y, \end{cases}$$

where κ is the Gaussian curvature of G . In fact the Riemannian manifold $(G, \langle \cdot, \cdot \rangle)$ together with the vector field X and expansion constant λ is a Ricci soliton if and only if the system (4.1) holds. Also we can see $X = \text{grad}\Phi$ if and only if

$$(4.2) \quad \begin{cases} \Phi_x = f\theta \\ \Phi_y = f\eta. \end{cases}$$

Example 4.1. If we let $f(x, y) = \frac{1}{1+x^2+y^2}$, $\lambda = 0$, $\theta = -2x$ and $\eta = -2y$ then we have the Hamilton's cigar which is a steady gradient Ricci soliton with Gaussian curvature $\kappa = \frac{2}{1+x^2+y^2}$ and potential function $\Phi(x, y) = -\ln(1+x^2+y^2)$ (one may want to use the equations (4.1), (4.2) and (3.7).).

Example 4.2. Suppose that $f(x, y) = \exp(x+y)$ and $X = \frac{\partial}{\partial x} + \frac{\partial}{\partial y}$. Then equations (4.1), (4.2) and (3.7) show that we have a flat shrinking gradient Ricci soliton with potential function $\Phi = f$ and expansion constant $\lambda = 1$.

Example 4.3. In the previous example if we consider $X = \frac{\partial}{\partial x} - \frac{\partial}{\partial y}$ then we have a flat steady Ricci soliton which is not gradient.

Remark 4.4. In two previous examples, one may want to work with g^{-1} or equivalently $f(x, y) = \exp(-x-y)$.

4.2. Lie group $G = \mathbb{R} \rtimes \mathbb{R}^+$. Now consider the Lie group $G = \mathbb{R} \rtimes \mathbb{R}^+$ equipped with a f -left invariant Riemannian metric g such that the set $\{E_1 := y\frac{\partial}{\partial y}, E_2 := y\frac{\partial}{\partial x}\}$ is an orthogonal basis at any point and is orthonormal at $e = (0, 1)$, where we have considered the natural coordinates (x, y) for $G = \mathbb{R} \rtimes \mathbb{R}^+$, $y > 0$. In this case we have $\alpha_{122} = -\alpha_{212} = 1$ and the other structural constants are zero. Suppose that $X = \theta E_1 + \eta E_2$ is an arbitrary vector field on G . Then the equation (1.1) shows that the Riemannian manifold $(G, \langle \cdot, \cdot \rangle)$ together with the vector field X and expansion constant λ is a Ricci soliton if and only if the following system holds,

$$(4.3) \quad \begin{cases} y\eta f_x + y\theta f_y + 2y\theta_y f = 2(\lambda - \kappa)f \\ y\eta f_x + y\theta f_y + 2f(y\eta_x - \theta) = 2(\lambda - \kappa)f \\ \eta = -y(\eta_y + \theta_x). \end{cases}$$

Also we can see $X = \text{grad}\Phi$ if and only if

$$(4.4) \quad \begin{cases} \Phi_x = \frac{f\eta}{y} \\ \Phi_y = \frac{f\theta}{y}. \end{cases}$$

Also by using equation (3.1) for Gaussian curvature of this manifold we have

$$(4.5) \quad \begin{aligned} \kappa &= \frac{1}{2f^3}(-f(f_{11} + f_{22}) + f_1^2 + f_2^2 + f f_1 - 2f^2) \\ &= \frac{1}{2f^3}(-f(y f_{yy} + y^2 f_{xx}) + y^2(f_x^2 + f_y^2) - 2f^2). \end{aligned}$$

Example 4.5. We start with left invariant Riemannian metric, in fact we consider $f(x, y) = 1$, for any $(x, y) \in G$. In this case, the equation (4.5) shows that this space is of constant Gaussian curvature $\kappa = -1$. Now if we consider $\theta = 0$, $\eta = \frac{1}{y}$ and $\lambda = -1$ then the system of equations (4.3) shows that G with the vector field $X = \frac{\partial}{\partial x}$ is an expanding Ricci soliton. The equations (4.4) show that this Ricci soliton is not gradient.

Example 4.6. Now consider the case that $f(x, y) = y$. In this case the equation (3.1) or (4.5) shows that the Gaussian curvature of G is $\kappa = -\frac{1}{2y}$. Suppose that $X = \frac{\partial}{\partial x} - \frac{\partial}{\partial y}$ (in fact let $\eta = -\theta = \frac{1}{y}$). Then G with the vector field X is a steady Ricci soliton which with attention to equations (4.4) is not gradient.

Example 4.7. In the previous example if we consider $X = -\frac{\partial}{\partial y}$ (or equivalently if $\theta = -\frac{1}{y}$ and $\eta = 0$) then by the systems (4.3) and (4.4) we have a gradient steady Ricci soliton with potential function $\Phi = \ln \frac{1}{y}$.

Example 4.8. Assume that $f(x, y) = y^2$ then we have a flat two dimensional noncommutative Lie group. If $X = \frac{\partial}{\partial x} + \frac{\partial}{\partial y}$ (equivalently $\eta = \theta = \frac{1}{y}$) then we have a steady gradient Ricci soliton with potential function $\Phi = x + y$.

Example 4.9. For $f(x, y) = y^2$ if $X = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}$ (equivalently $\eta = \frac{x}{y}$ and $\theta = 1$) then we have a flat two dimensional shrinking gradient Ricci soliton with $\lambda = 1$ and potential function $\Phi = \frac{1}{2}(x^2 + y^2)$.

Example 4.10. In two previous examples if $X = \frac{\partial}{\partial y}$ (equivalently $\eta = 0$ and $\theta = \frac{1}{y}$) then we have a flat two dimensional steady gradient Ricci soliton with potential function $\Phi = y + c$, where c is an arbitrary constant real number.

4.3. Lie group $G = \mathbb{R}^2 \rtimes \mathbb{R}^+$. In this subsection we consider the Lie group $G = \mathbb{R} \rtimes \mathbb{R}^+$ with natural coordinates (x, y, z) such that $z > 0$. Similar to above we consider a f -left invariant Riemannian metric $\langle \cdot, \cdot \rangle$ such that the set $\{E_1 := z\frac{\partial}{\partial z}, E_2 := z\frac{\partial}{\partial x}, E_3 := z\frac{\partial}{\partial y}\}$ is an orthogonal basis at any point and is orthonormal at $e = (0, 0, 1)$. Easily we can see $\alpha_{122} = \alpha_{133} = 1$, $\alpha_{212} = \alpha_{313} = -1$ and the other structural constants are zero. Suppose that $X = \theta E_1 + \eta E_2 + \mu E_3$ is an arbitrary vector field on G . Then the equations (1.1) together with (3.8) show that the Riemannian manifold $(G, \langle \cdot, \cdot \rangle)$ with the vector field X and expansion constant λ is a Ricci soliton if and only if the following system holds,

$$(4.6) \quad \begin{cases} X(f) + 2\theta_1 f = 2\left(\lambda f - \frac{1}{4f^2}(-2f(2f_{11} + f_{22} + f_{33}) + 4f_1^2 + f_2^2 + f_3^2 + 4f(f_1 - 2f))\right) \\ f(\eta + \eta_1 + \theta_2) = -\frac{1}{2f^2}(3f_1 f_2 - 2f f_{21}) \\ f(\mu + \mu_1 + \theta_3) = -\frac{1}{2f^2}(3f_1 f_3 - 2f f_{31}) \\ X(f) + 2f(\eta_2 - \theta) = 2\left(\lambda f - \frac{1}{4f^2}(-2f(f_{11} + 2f_{22} + f_{33}) + f_1^2 + 4f_2^2 + f_3^2 + 2f(3f_1 - 4f))\right) \\ f(\mu_2 + \eta_3) = -\frac{1}{2f^2}(3f_2 f_3 - 2f f_{32}) \\ X(f) + 2f(\mu_3 - \theta) = 2\left(\lambda f - \frac{1}{4f^2}(-2f(f_{11} + f_{22} + 2f_{33}) + f_1^2 + f_2^2 + 4f_3^2 + 2f(3f_1 - 4f))\right). \end{cases}$$

A simple computation shows that $X = \text{grad}\Phi$ if and only if

$$(4.7) \quad \begin{cases} \Phi_x = \frac{f\eta}{z} \\ \Phi_y = \frac{f\mu}{z} \\ \Phi_z = \frac{f\theta}{z}. \end{cases}$$

Example 4.11. In a special case for left invariant Riemannian metric induced by $f(x, y, z) = 1$, if $X = \frac{\partial}{\partial x} + \frac{\partial}{\partial y}$ (equivalently $\eta = \mu = \frac{1}{z}$ and $\theta = 0$) then by using the systems (4.6) and (4.7) and the equation (3.1) we have a three dimensional expanding Ricci soliton with $\lambda = -2$ which is not gradient. Also by using (3.1) for its sectional curvature we have $K(E_1, E_2) = K(E_1, E_3) = K(E_2, E_3) = -1$.

Example 4.12. In the previous example if $X = \frac{\partial}{\partial x}$ (equivalently $\eta = \frac{1}{z}$ and $\theta = \mu = 0$) then we have a three dimensional expanding Ricci soliton with $\lambda = -2$ which is not gradient.

Example 4.13. In two previous examples if $X = \frac{\partial}{\partial y}$ (equivalently $\mu = \frac{1}{z}$ and $\theta = \eta = 0$) again we have a three dimensional expanding Ricci soliton with $\lambda = -2$ which is not gradient.

Example 4.14. Consider the f -left invariant Riemannian metric induced by $f(x, y, z) = z^2$, if $X = \frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}$ (equivalently $\theta = \eta = \mu = \frac{1}{z}$) then the systems (4.6) and (4.7) together with (3.1) show that the Riamnnian manifold G with the vector field X is a flat three dimensional steady gradient Ricci soliton with potential function $\Phi = x + y + z$.

4.4. Lie group $G = \mathbb{R} \rtimes \mathbb{R}^+ \times \mathbb{R}$. Now we consider another noncommutative Lie group $G = \mathbb{R} \rtimes \mathbb{R}^+ \times \mathbb{R}$ with natural coordinates (x, y, z) such that $y > 0$. Then consider a f -left invariant Riemannian metric $\langle \cdot, \cdot \rangle$ such that the set $\{E_1 := y\frac{\partial}{\partial y}, E_2 := y\frac{\partial}{\partial x}, E_3 := \frac{\partial}{\partial z}\}$ is an orthogonal basis at any point and is orthonormal at $e = (0, 1, 0)$. In this case we have $\alpha_{122} = -\alpha_{212} = 1$ and the other structural constants are zero. Suppose that $X = \theta E_1 + \eta E_2 + \mu E_3$ is an arbitrary vector field on G . Then the equations (1.1) and (3.8) show that the Riemannian manifold $(G, \langle \cdot, \cdot \rangle)$ with the vector field X and expansion constant λ is a Ricci soliton if and only if the following system holds,

$$(4.8) \quad \begin{cases} X(f) + 2\theta_1 f = 2\left(\lambda f - \frac{1}{4f^2}(-2f(2f_{11} + f_{22} + f_{33}) + 4f_1^2 + f_2^2 + f_3^2 + 2f(f_1 - 2f))\right) \\ f(\eta + \eta_1 + \theta_2) = -\frac{1}{2f^2}(3f_1 f_2 - 2f f_{21}) \\ f(\mu_1 + \theta_3) = -\frac{1}{2f^2}(3f_1 f_3 - 2f f_{31}) \\ X(f) + 2f(\eta_2 - \theta) = 2\left(\lambda f - \frac{1}{4f^2}(-2f(f_{11} + 2f_{22} + f_{33}) + f_1^2 + 4f_2^2 + f_3^2 + 4f f_1 - 4f^2)\right) \\ f(\mu_2 + \eta_3) = -\frac{1}{2f^2}(3f_2 f_3 - 2f f_{32}) \\ X(f) + 2\mu_3 f = 2\left(\lambda f - \frac{1}{4f^2}(-2f(f_{11} + f_{22} + 2f_{33}) + f_1^2 + f_2^2 + 4f_3^2 + 2f f_1)\right). \end{cases}$$

Also we can see $X = \text{grad}\Phi$ if and only if

$$(4.9) \quad \begin{cases} \Phi_x = \frac{f\eta}{y} \\ \Phi_y = \frac{f\theta}{y} \\ \Phi_z = \mu f. \end{cases}$$

Example 4.15. If $f(x, y, z) = 1$ then we have $K(E_1, E_2) = -1$, $K(E_1, E_3) = K(E_2, E_3) = 0$ and the only non zero $\text{Ric}(E_i, E_j)$'s are $\text{Ric}(E_1, E_1) = \text{Ric}(E_2, E_2) = -1$. Suppose that $X = -z\frac{\partial}{\partial z}$ (equivalently $\theta = \eta = 0$ and $\mu = -z$) then by using the systems (4.8) and (4.9) we can see $(G, \langle \cdot, \cdot \rangle)$ with X is a three dimensional expanding gradient Ricci soliton with $\lambda = -1$ and potential function $\Phi = -\frac{1}{2}z^2$.

Example 4.16. In the previous example if we consider $X = \frac{\partial}{\partial x} - z\frac{\partial}{\partial z}$ (equivalently $\theta = 0$, $\eta = \frac{1}{y}$ and $\mu = -z$) then we have a three dimensional expanding Ricci soliton with $\lambda = -1$ which is not gradient.

Example 4.17. Assume that $f(x, y, z) = y^2$ then the equation (3.1) shows that $(G, \langle \cdot, \cdot \rangle)$ is flat. Suppose that $X = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}$ (equivalently $\theta = 1$, $\eta = \frac{x}{y}$ and $\mu = 0$) then the systems (4.8) and (4.9) show that $(G, \langle \cdot, \cdot \rangle)$ with X is a three dimensional shrinking gradient Ricci soliton with $\lambda = 1$ and potential function $\Phi = \frac{1}{2}(x^2 + y^2)$.

4.5. Lie group $G = \mathbb{R} \rtimes \mathbb{R}^+ \times \mathbb{R}^2$. In this subsection we consider the noncommutative four dimensional Lie group $G = \mathbb{R} \rtimes \mathbb{R}^+ \times \mathbb{R}^2$ with natural coordinates (x, y, z, w) such that $y > 0$. Now suppose that $\langle \cdot, \cdot \rangle$ is a f -left invariant Riemannian metric on G such that the set $\{E_1 := y\frac{\partial}{\partial y}, E_2 := y\frac{\partial}{\partial x}, E_3 := \frac{\partial}{\partial z}, E_4 := \frac{\partial}{\partial w}\}$ is an orthogonal basis at any point and is orthonormal at $e = (0, 1, 0, 0)$. For this Lie group we have $\alpha_{122} = -\alpha_{212} = 1$ and the other structural constants are zero. Assume that $X = \theta E_1 + \eta E_2 + \mu E_3 + \nu E_4$ is an arbitrary vector field on G . Then the equations (1.1) and (3.8) show that the Riemannian manifold $(G, \langle \cdot, \cdot \rangle)$ with the vector field X and expansion constant λ is a Ricci soliton if and only if the following system holds,

$$(4.10) \quad \begin{cases} X(f) + 2\theta_1 f = 2\left(\lambda f - \frac{1}{2f^2}(-f(3f_{11} + f_{22} + f_{33} + f_{44}) + 3f_1^2 + f(f_1 - 2f))\right) \\ f(\eta + \eta_1 + \theta_2) = -\frac{1}{f^2}(3f_1 f_2 - 2f f_{21}) \\ f(\mu_1 + \theta_3) = -\frac{1}{f^2}(3f_1 f_3 - 2f f_{31}) \\ f(\nu_1 + \theta_4) = -\frac{1}{f^2}(3f_1 f_4 - 2f f_{41}) \\ X(f) + 2f(\eta_2 - \theta) = 2\left(\lambda f - \frac{1}{2f^2}(-f(f_{11} + 3f_{22} + f_{33} + f_{44}) + 3f_2^2 + f(3f_1 - 2f))\right) \\ f(\mu_2 + \eta_3) = -\frac{1}{f^2}(3f_2 f_3 - 2f f_{32}) \\ f(\nu_2 + \eta_4) = -\frac{1}{f^2}(3f_2 f_4 - 2f f_{42}) \\ X(f) + 2\mu_3 f = 2\left(\lambda f - \frac{1}{2f^2}(-f(f_{11} + f_{22} + 3f_{33} + f_{44}) + 3f_3^2 + f f_1)\right) \\ f(\nu_3 + \mu_4) = -\frac{1}{f^2}(3f_3 f_4 - 2f f_{43}) \\ X(f) + 2\nu_4 f = 2\left(\lambda f - \frac{1}{2f^2}(-f(f_{11} + f_{22} + f_{33} + 3f_{44}) + 3f_4^2 + f f_1)\right). \end{cases}$$

Also we can see $X = \text{grad}\Phi$ if and only if

$$(4.11) \quad \begin{cases} \Phi_x = \frac{f\eta}{y} \\ \Phi_y = \frac{f\theta}{y} \\ \Phi_z = \mu f \\ \Phi_w = \nu f. \end{cases}$$

Example 4.18. Suppose that the left invariant Riemannian metric $\langle \cdot, \cdot \rangle$ generated by $f(x, y, z, w) = 1$ on G . Then equation (3.1) shows that $K(E_1, E_2) = -1$ and $K(E_i, E_j) = 0$ in other cases. Now consider $X = -z\frac{\partial}{\partial z} - w\frac{\partial}{\partial w}$ (in fact we assumed that $\theta = \eta = 0$, $\mu = -z$ and $\nu = -w$). The systems (4.10) and (4.11) show that $(G, \langle \cdot, \cdot \rangle)$ with X is an expanding gradient Ricci soliton with $\lambda = -1$ and potential function $\Phi = -\frac{1}{2}(z^2 + w^2)$.

Example 4.19. In the previous example if we suppose that $X = \frac{\partial}{\partial x} - z\frac{\partial}{\partial z} - w\frac{\partial}{\partial w}$ (equivalently $\theta = 0$, $\eta = \frac{1}{y}$, $\mu = -z$ and $\nu = -w$) then we have an expanding Ricci soliton with $\lambda = -1$ which is not gradient.

4.6. Lie group $G = \mathbb{R} \rtimes \mathbb{R}^+ \times \mathbb{R} \rtimes \mathbb{R}^+$. Now we give another examples of four dimensional Ricci solitons. Suppose that $G = \mathbb{R} \rtimes \mathbb{R}^+ \times \mathbb{R} \rtimes \mathbb{R}^+$ and consider the natural coordinates (x, y, z, w) , such that $y > 0$ and $w > 0$, for it. Consider the f -left invariant Riemannian

metric $\langle \cdot, \cdot \rangle$ on G such that the set $\{E_1 := y \frac{\partial}{\partial y}, E_2 := y \frac{\partial}{\partial x}, E_3 := w \frac{\partial}{\partial w}, E_4 := w \frac{\partial}{\partial z}\}$ is an orthogonal basis at any point and is orthonormal at $e = (0, 1, 0, 1)$. Easily we can see $\alpha_{122} = -\alpha_{212} = \alpha_{344} = -\alpha_{434} = 1$ and the other structural constants are zero. As before let $X = \theta E_1 + \eta E_2 + \mu E_3 + \nu E_4$ be any vector field on G . The equations (1.1) and (3.8) show that the Riemannian manifold $(G, \langle \cdot, \cdot \rangle)$ with the vector field X and expansion constant λ is a Ricci soliton if and only if the following system holds,

$$(4.12) \quad \begin{cases} X(f) + 2\theta_1 f = 2 \left(\lambda f - \frac{1}{2f^2} (-f(3f_{11} + f_{22} + f_{33} + f_{44}) + 3f_1^2 + f(f_1 + f_3 - 2f)) \right) \\ f(\eta + \eta_1 + \theta_2) = -\frac{1}{f^2} (3f_1 f_2 - 2f f_{21}) \\ f(\mu_1 + \theta_3) = -\frac{1}{f^2} (3f_1 f_3 - 2f f_{31}) \\ f(\nu_1 + \theta_4) = -\frac{1}{f^2} (3f_1 f_4 - 2f f_{41}) \\ X(f) + 2f(\eta_2 - \theta) = 2 \left(\lambda f - \frac{1}{2f^2} (-f(f_{11} + 3f_{22} + f_{33} + f_{44}) + 3f_2^2 + f(3f_1 + f_3 - 2f)) \right) \\ f(\mu_2 + \eta_3) = -\frac{1}{f^2} (3f_2 f_3 - 2f f_{32}) \\ f(\nu_2 + \eta_4) = -\frac{1}{f^2} (3f_2 f_4 - 2f f_{42}) \\ X(f) + 2\mu_3 f = 2 \left(\lambda f - \frac{1}{2f^2} (-f(f_{11} + f_{22} + 3f_{33} + f_{44}) + 3f_3^2 + f(f_1 + f_3 - 2f)) \right) \\ f(\nu + \nu_3 + \mu_4) = -\frac{1}{f^2} (3f_3 f_4 - 2f f_{43}) \\ X(f) + 2f(\nu_4 - \mu) = 2 \left(\lambda f - \frac{1}{2f^2} (-f(f_{11} + f_{22} + f_{33} + 3f_{44}) + 3f_4^2 + f(f_1 + 3f_3 - 2f)) \right). \end{cases}$$

Also $X = \text{grad}\Phi$ if and only if

$$(4.13) \quad \begin{cases} \Phi_x = \frac{f\eta}{y} \\ \Phi_y = \frac{f\theta}{y} \\ \Phi_z = \frac{f\nu}{w} \\ \Phi_w = \frac{f\mu}{w}. \end{cases}$$

Example 4.20. Consider the left invariant Riemannian metric $\langle \cdot, \cdot \rangle$ induced by the constant function $f(x, y, z, w) = 1$ on G . Then by (3.1) for the sectional curvature we have $K(E_1, E_2) = K(E_3, E_4) = -1$ and $K(E_i, E_j) = 0$ in other cases. Now suppose that $X = x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y}$ ($\theta = 1$, $\eta = \frac{x}{y}$, $\mu = \nu = 0$). The systems (4.12) and (4.13) show that $(G, \langle \cdot, \cdot \rangle)$ with X is an expanding Ricci soliton with $\lambda = -1$ which is not gradient.

Remark 4.21. In the previous example one may want to work with $X = z \frac{\partial}{\partial z} + w \frac{\partial}{\partial w}$.

Remark 4.22. Note that one can use the above equations to construct Ricci almost solitons (for more detail about Ricci almost solitons see [12]). For example in subsection 4.1 if we let $f(x, y) = \exp(x^2 + y^2)$ and $X = -y \frac{\partial}{\partial x} + x \frac{\partial}{\partial y}$, then equations (4.1), (4.2) and (3.7) show that we have a nontrivial shrinking Ricci almost soliton with expansion function $\lambda(x, y) = 2 \exp(-x^2 - y^2)$ and Gaussian curvature $-2 \leq \kappa = -2 \exp(-x^2 - y^2) < 0$ which is not gradient. As another example in the example (4.5) if we suppose that $\theta = \frac{1}{y}$ and $\eta = 0$ then we have an indefinit gradient Ricci almost soliton with expansion function $\lambda(x, y) = \frac{-1-y}{y}$ and potential function $\Phi = -\frac{1}{y}$.

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